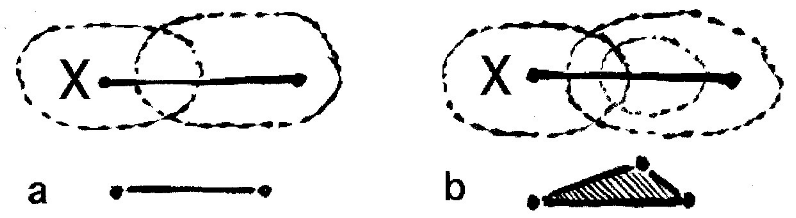

Figure 5: Nerve of covering III

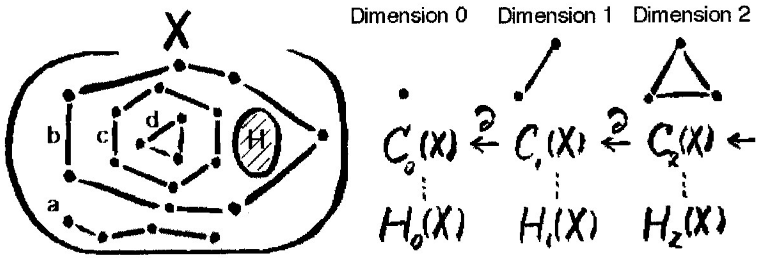

Figure 6: H is a hole in 2-dimensional space X. a, b, c, and d are 1-dimensional chains. b, c, and d are 1-dimensional cycles. c and d are in the same equivalence class of 1-cycles, but not b

represents the connectivity of space X through an equivalence class of q-dimensional *cycles*, i.e., a closed *chain* formed by q-dimensional simplexes in X (see Figure 6). The equivalence class is defined by regarding two q-dimensional cycles as equal if their difference can be identified with the *boundary* of a certain $(q+1)$-dimensional chain in X. (In Figure 6, the special feature, hole H in space X, is expressed by the equivalence class of 1-cycle b.) For the present, suppose that such $H_q(X)$'s, $q = 0, 1, 2, \ldots$, are vector spaces over real field R and the algebraic structure on each of them successfully stands for the above intuitive discussion about chain formation. The series of such a *graded* vector space

$$\cdots \to H_{q+1}(X) \to H_q(X) \to H_{q-1} \to \cdots \to H_1(X) \to H_0(X) \to 0 \tag{1.2}$$

describes all of the necessary features of space X, where arrows represent the canonical linear functions determined by the concept to take the *boundary* of the chains and 0 denotes the vector space of $\{0\}$. If X is a single point, each q-th homology group is 0 except for $H_0(X) \simeq R$. The acyclic sets

are those whose homology groups are exactly the same as those of a single point.

Since there is essentially no difference between acyclic sets and single points, it is not surprising that under homology theory we obtain fixed-point theorems for acyclic valued mappings. *Eilenberg–Montgomery's fixed-point theorem*, one of the most basic results for such multivalued mappings, is far more general than theorems for convex valued mappings, as long as we permit conditions on the space that enable us to use such homological arguments as Čech theory (e.g., polyhedron, absolute neighborhood retracts, and the *local connectedness condition*). Such an identification of sets and points also plays an essential role in relating our convex and topological arguments to algebraic ones (e.g., there is a standard method for constructing associated algebraic mappings, the *method of acyclic models*) and in presenting basic theorems for the settings of Čech-type homology theory (e.g., the *Vietoris mapping theorem*).

The methods and approaches in Chapters 2–5 may be summarized as the replacement of continuity and/or convexity conditions for equilibrium and fixed-point theorems by weaker conditions using the directions of points defined by the dual-system structure. Chapters 6 and 7 show their relationship to the algebraic properties described under homological theory. Moreover, one can see how the underlying ideas in earlier chapters are also utilized for further progress in the arguments, even in algebraic homological settings.

As noted before, the use of algebraic topology rather than differential topology is related to the main purpose of this book: to list *minimal* logical, set-theoretic, and algebraic *requirements* for economic equilibrium and its closely related arguments. The point will also be summarized in Chapters 9 and 10, where all basic features and discussions before Chapter 9 are reexamined through finitistic and recursive methods in an axiomatic set theory.

1.2.4 *Axiomatic set theory*

Chapter 9 refers to the foundation of mathematics as a basic tool or a language for describing a theory of social sciences. Since one important purpose of social science is to describe human society as a well-founded and -defined entity, or a *model*, a formal treatment of our basic tool of thought (mathematics) itself is critical to formalize the foundation of our knowledge.