Mathematically, all arguments in this book are based on *ZF*, the Zermelo–Fraenkel set theory (the most standard axiomatic set theory) written by first-order predicate logic (one of the most popular formal languages). Such a framework, as our basic standpoint, is necessary because our basic theory must be sufficiently strong to incorporate not only our ordinary mathematical arguments but also all necessary procedures in describing the theory itself as formal objects. Indeed, the list of axioms in Zermelo–Fraenkel set theory, which can be used to develop almost all of our ordinary mathematics, is also simple enough to be characterized by standard finitistic or recursive methods that are obviously incorporated in *ZF*.

Of course, until Chapter 9, readers need not be concerned about what axioms our basic theory depends on. The basic mathematical concepts and methods in this book (introduced in the next section) presume a merely natural and naive interpretation of ordinary language. Note, however, that such set-theoretic axioms and their finitistic or recursive methods are not special concepts for a certain field of mathematics, but rather relate to the one thing that never changes in our development of knowledge: the linguistic feature of mathematics.

1.3 Basic Mathematical Concepts and Definitions

The mathematical concepts and definitions that are necessary but not immediately connected with this book's fixed-point or economic equilibrium arguments are gathered into three parts: this section for general fundamental notions, the first section of Chapter 6 for an introduction to algebraic topology, and the first section of Chapter 9 for concepts in axiomatic set theory and mathematical logic. The main purpose of these sections is merely to give definitions of mathematical terms.

In principle, all the concepts and theorems in this book can be explained without any presupposed notion in mathematics and are completely supported in the book. Consequently, all the mathematical topics could be arranged from the basic to the advanced ones so that no theorem is used to prove other results before its own proof is presented. Such an attempt, however, would almost certainly force readers to study several boring mathematical textbooks before reaching the special topics of this book that are not necessarily based on mathematical details and proofs. Therefore, throughout this book, several mathematical theorems and properties are treated (at first) as given, and their proofs are given in later chapters.

Moreover, to facilitate the descriptions of ordinary notions in mathematical economics in Chapters 2–5, readers are expected at least to have a basic knowledge of Euclidean spaces equivalent to college freshman calculus and linear algebra.

In this section, with the definitions of mathematical terms in elementary topology (Subsection 1.3.1), I will introduce two important basic theorems: the partition of unity theorem (Subsection 1.3.2) and the separation hyperplane theorem (Subsection 1.3.3).[9]

1.3.1 *Sets, topologies, and notational conventions*

All the mathematical arguments in this book are based on Zermelo–Fraenkel set theory with Axiom of Choice, written by first-order predicate calculus. As stated before, these comprise one of the most common pairs of an axiomatic theory of sets and a basic formal language. I merely note here that the following chapters are based on a very standard foundation of mathematics. The formal treatments of the axiomatic set theory and formal language are given in Chapter 9.[10] We also use the notions of Bourbaki (1939) (e.g., structures and inverse and direct limits) in later chapters and Kelley (1955) (many definitions in topology), as long as the underlying set-theoretic differences are not significant.

Sets

Theory of sets is a theory that has only two predicates, $\in$ and $=$, elementhood and equality. We often denote a *set* by the form $\{x|P(x)\}$, where $P(x)$ denotes a property of x described under our formal language (first-order predicate calculus). Notation $\{x|P(x)\}$ represents the class of objects having property P which, in some cases, may not be treated as a proper mathematical object or a *set*. The axioms of the theory of sets (e.g., *ZF* with Axiom of Choice) give rules for a property under which class $\{x|P(x)\}$ may be called a *set*. (A careless use of such properties may cause

[9] In this book, adding to these two theorems, Brouwer's fixed-point theorem (in Chapter 2) will be introduced and repeatedly used before its proof is presented. The proof of Brouwer's fixed-point theorem is given in Chapter 6. Proofs for other theorems are given in Mathematical Appendices I and II.

[10] For references, see also Fraenkel *et al.* (1973), Kunen (1980), Jech (1997), etc.

problems like the well-known Russell Paradox.[11]) For example, the class of all natural numbers, $N = \{0, 1, 2, \ldots\}$, the family of two sets x and y, $\{x, y\}$, the *ordered pair* of two sets x and y, (x, y), the class of all subsets of set A (the *power set* of A), $\mathscr{P}(A) = \{X | X \subset A\}$, and unions and products for the family of sets (see below) are assured to be sets under the axioms of ZF.

Family of sets

For family (set of sets) $\mathscr{U}$, we denote by $\bigcup \mathscr{U}$ the *union* of elements of $\mathscr{U}$. If the elements of family $\mathscr{U}$ are indexed by set I as $\mathscr{U} = \{U_i | i \in I\}$, we often write $\bigcup_{i \in I} U_i$ instead of $\bigcup \mathscr{U}$. If family $\mathscr{U} = \{U_i | i \in I\}$ is not empty, we denote by $\bigcap \mathscr{U}$ or $\bigcap_{i \in I} U_i$ the *intersection* of elements of $\mathscr{U}$. Denote by $A \setminus B$ the *set-theoretic difference* between two sets A and B, i.e., $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$. Given set X and non-empty family $\{U_i | i \in I\}$, the following important relations hold among unions, intersections, and differences: $X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I}(X \setminus U_i)$ and $X \setminus \bigcap_{i \in I} U_i = \bigcup_{i \in I}(X \setminus U_i)$ (De Morgan's laws).

Cartesian products and relations

Given two sets, X and Y, the *Cartesian product* (or *direct product*) $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A *relation* is a set of ordered pairs. A subset of Cartesian product $X \times Y$ of X and Y is called a *relation on X to Y*. For relation φ, the *domain* of φ is the set $\mathrm{dom}\,(\varphi) = \{x | \exists y, (x, y) \in \varphi\}$, and the *range* of φ is the set $\mathrm{ran}\,(\varphi) = \{y | \exists x, (x, y) \in \varphi\}$. If φ and ψ are relations, the *composition* of φ and ψ is the relation $\zeta = \{(x, z) | \exists y, (x, y) \in \varphi \text{ and } (y, z) \in \psi\}$, and ζ is denoted by $\psi \circ \varphi$. For relation φ on X to Y, the *upper section* of φ at $x \in X$ (*x-section of φ*) is the set $\{y | (x, y) \in \varphi\}$, which is denoted by $\varphi(x)$. Similarly, the *lower section* of φ at $y \in Y$ is the set $\{x | (x, y) \in \varphi\}$. We define φ^{-1} for relation φ as $\varphi^{-1} = \{(x, y) | (y, x) \in \varphi\}$. Then, the lower section of φ at $y \in \mathrm{ran}\,(\varphi)$ is nothing but $\varphi^{-1}(y)$, which is the upper section of φ^{-1} at y.

[11] Let $T = \{x | x \notin x\}$. Consider whether T is an element of T. If $T \in T$, then by the definition of T, we have $T \notin T$, a contradiction. Hence, we have a proof for $T \notin T$. On the other hand, $T \notin T$ implies that T satisfies the sufficient condition for an element of T. Therefore, we have also a proof for $T \in T$. It follows that for the consistency of the theory, we cannot treat such T as a set (object) in the domain of discourse.

For two relations φ and ψ, φ is a *restriction* of ψ if $\mathrm{dom}\,(\varphi) \subset \mathrm{dom}\,(\psi)$ and $\varphi(x) = \psi(x)$ for all $x \in \mathrm{dom}\,(\varphi)$, and ψ is an *extension* of φ if $\varphi \subset \psi$.

Functions and correspondences

A *function* f on X to Y, denoted by $f : X \to Y$, is a relation on X to Y such that $\mathrm{dom}\,(f) = X$ and every upper section is a singleton. Function φ on X to 2^Y, where 2^Y denotes the family of all subsets of Y, is called a *correspondence* on X to Y and is also denoted by $\varphi : X \to Y$ or, more precisely, by $\varphi : X \ni x \mapsto \varphi(x) \subset Y$. For function f on X to Y, the unique element of the upper section (not the singleton itself) at x is traditionally denoted by $f(x)$, so we write $f : X \to Y$ and $f : X \ni x \mapsto f(x) \in Y$. Element $f(x)$ is also called the *image* of x under f. On the other hand, the lower section of f at $y \in Y$, $f^{-1}(y)$, itself is called the *inverse image* of y under f. Function $f : X \to Y$ is said to be *injective* (*one to one*) if for all x and x' in X, $x \neq x'$ means $f(x) \neq f(x')$ and is said to be *surjective* (*onto*) if for all $y \in Y$ there is element $x \in X$ such that $y = f(x)$. Two sets X and Y are said to have the same *cardinality* if there is a *bijective* (injective and surjective) function $f : X \to Y$. A set having the same cardinality with a subset of $N = \{0, 1, 2, \ldots\}$ is called a *countable set*.

Binary relations

A *binary relation* on X is a subset of $X \times X$. For binary relation $\mathscr{R} \subset X \times X$, we customarily write $x\mathscr{R}y$ instead of $(x, y) \in \mathscr{R}$. Binary relation $\mathscr{R}$ on X is said to be a *preordering* if it is *reflexive* ($\forall x \in X, x\mathscr{R}x$) and *transitive* ($\forall x, y, z \in X$, ($x\mathscr{R}y$ and $y\mathscr{R}z$) $\Longrightarrow x\mathscr{R}z$). The pair of X and preordering $\mathscr{R}$ on X, $(X, \mathscr{R})$, is called a *preordered set*. A *directed* set is a non-empty preordered set such that for each of its elements i, j, element k satisfies $k\mathscr{R}i$ and $k\mathscr{R}j$. Preordering $\mathscr{R}$ on X is said to be an *ordering* if it is *antisymmetric* ($\forall x, y \in X$, ($x\mathscr{R}y$ and $y\mathscr{R}x$) $\Longrightarrow x = y$). If preordering $\mathscr{R}$ on X is *symmetric* ($\forall x, y \in X$, ($x\mathscr{R}y$) $\Longrightarrow y\mathscr{R}x$), it is called an *equivalence relation* on X. Given two preordered sets $(X, \mathscr{R})$ and $(Y, \mathscr{Q})$, mapping $f : X \to Y$ is said to be *monotone* (*isotone, order preserving*) if $x\mathscr{R}z$ implies $f(x)\mathscr{Q}f(z)$ for each $x, z \in X$.

Axiom of choice and products of a family of sets

Given family (set) of sets $\{X_i | i \in I\}$, the *Cartesian product* of the family of sets, $\prod_{i \in I} X_i$, is the set of functions on I to $\bigcup_{i \in I} X_i$ such that for each

$i \in I$, the image of i, x_i, belongs to X_i. Such a function, $f : I \to \bigcup_{i \in I} X_i$, is called a *choice function.* The existence of at least one choice function for each non-empty family of non-empty sets is assured in the theory of sets as an axiom called the *Axiom of Choice.* If there is binary relation $\mathscr{Q}_i$ on X_i for each $i \in I$, we may naturally define the *product relation* $\mathscr{Q}$ on $X = \prod_{i \in I} X_i$ as $f \mathscr{Q} g$ if and only if $f(i) \mathscr{Q}_i g(i)$ for all $i \in I$. Product relation $\mathscr{Q}$ is reflexive, transitive, and anti-symmetric if all $\mathscr{Q}_i$s are reflexive, transitive, and anti-symmetric respectively. Hence, $(X, \mathscr{Q})$ is a directed, preordered, and ordered set as long as all component spaces are directed, preordered, and ordered sets respectively, where for the non-emptiness of X and Q, the choice axiom is necessary.

Topology

A *topology* on space (set) X is a family of subsets of X, $\mathscr{T}$, satisfying the conditions that (1) $X \in \mathscr{T}$, (2) $\emptyset \in \mathscr{T}$, (3) for each non-empty finite subset $\mathscr{U} \subset \mathscr{T}$, the intersection $\bigcap \mathscr{U} = \bigcap_{U \in \mathscr{U}} U$ is an element of $\mathscr{T}$, and (4) for each subset $\mathscr{U} \subset \mathscr{T}$, the union $\bigcup \mathscr{U} = \bigcup_{U \in \mathscr{U}} U$ is an element of $\mathscr{T}$. Pair $(X, \mathscr{T})$ is called a *topological space*, and each element $U \in \mathscr{T}$ is said to be an *open* set in topological space $(X, \mathscr{T})$. The complement of an open set, $X \setminus U$, $U \in \mathscr{T}$, is called a *closed* set. For each point x in a topological space, set V including open set $U \ni x$ is called a *neighborhood* of x. For subset A of topological space $(X, \mathscr{T})$, we define the *relativization* $\mathscr{T}_A$ *of* $\mathscr{T}$ *on* A as $\mathscr{T}_A = \{U \cap A | U \in \mathscr{T}\}$. (Verify that $\mathscr{T}_A$ is a topology on A.)

Closure and interior

By the definition of topology, it is clear that (1) $\emptyset$ is closed, (2) total space X is closed, (3) the finite union of closed sets is closed, and (4) an arbitrary intersection of closed sets is closed. For subset A of topological space X, therefore, we may define the smallest closed set containing A, the *closure* of A, as $\operatorname{cl} A = \bigcap\{B | A \subset B, B \text{ is closed in } X\}$. In the same way, we may define the largest open set contained in A, the *interior* of A, as $\operatorname{int} A = \bigcup\{B | B \subset A, B \text{ is open in } X\}$.

Continuity

If $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces, function $f : X \to Y$ is *continuous* if, for each open set $U_Y \in \mathscr{T}_Y$, the *inverse image of set* U_Y, $f^{-1}(U_Y) = \{x \in X | f(x) \in U_Y\}$, is an element of $\mathscr{T}_X$. The condition is equivalent to saying that at each $x \in X$, for every open set $U \ni f(x)$, there is open set $V \ni x$ such that the *image of set* V, $f(V) = \{f(z) | z \in V\}$, is a subset of U. (Since a set is open iff for each of its elements there is an open neighborhood contained in the set, the latter condition is sufficient for the former. For the necessity, use the property that $f(f^{-1}(U)) \subset U$ for any $U \subset Y$.) It is easy to see that if $f : X \to Y$ and $g : Y \to Z$ are continuous, then their composition $g \circ f$ is also continuous.

Convergence

A *net* in topological space X is a function $S : D \to X$ whose domain $(D, \geq)$ is a directed set. If D is the set of all natural numbers with the ordinary $\geq$ relation, net is called a *sequence.* Net S in X *converges* to $x^* \in X$, if for each neighborhood U of x^* there exists $\bar{\nu} \in D$ such that $\forall \nu \geq \bar{\nu}$, $S(\nu) \in U$. (Net S is said to be *eventually in* U.) Net (also called a *generalized sequence*) is a useful concept to describe closedness, continuity of mappings, etc., for general topological spaces in exactly the same way as the notion of convergent sequence does in Euclidean spaces. One can verify that set $A \subset X$ is closed if and only if for every net in A converging to a certain point x in X, $x \in A$ necessarily follows. Furthermore, we may prove that function $f : X \to Y$ is continuous if and only if for every net $S : D \to X$ on X, net $f \circ S : D \to Y$ converges to $f(x^*) \in Y$ as long as S converges to $x^* \in X$. (Use the second condition, $\forall U \ni f(x), \exists V \ni x, f(V) \subset U$, for the continuity. The necessity of this third net-characterization condition is trivial. For the sufficiency, define net S on the directed set of neighborhoods of $x^* \in X$ at which the second condition for the continuity is not satisfied.)

Subnet and cluster point

A *subnet* of net $S : D \to X$ is net $T : E \to X$ such that mapping M exists on directed set E to D satisfying $T = S \circ M$ and the condition that for all $m \in D$ element $\bar{n} \in E$ exists such that $M(n) \geq m$ for all $n \geq \bar{n}$. The condition is typically satisfied when M is monotone and for all $m \in D$ element $n \in E$ exists such that $M(n) \geq m$. (More specifically, when E is a subset of D such that for all $m \in D$ there is an element $n \in E$, i.e., E is a *cofinal* subset of D. Although this may seem a standard way of constructing subnets, such a simple class of subnets is not sufficient for all purposes, unfortunately.) For net $S : D \to X$, point $x \in X$ is called a *cluster point* of S if for all neighborhoods U of x, for all $\bar{\nu}$ in D, there is $\nu \geq \bar{\nu}$ such that $S(\nu) \in U$. (Net S is said to be *frequently in* U.) One may prove that if x is a cluster point of net $S : D \to X$, then there is a subnet of S converging to x. (To see this, let $\mathscr{N}$ be the set of all open neighborhoods of x directed by the inclusion, and for each $N \in \mathscr{N}$ let D_N be the cofinal subset of D such that $S(\nu) \in N$ for all $\nu \in D_N$. Consider mapping $M : \mathscr{N} \times \prod_{N \in \mathscr{N}} D_N \ni (N, f) \mapsto f(N) \in D$ on the product directed set and subnet $T = S \circ M$. Or let $E \subset D \times \mathscr{N}$ be the set of all pairs (d, N) such that $S(d) \in N$ under product ordering, define M on E to D as $M(d, N) = d$, and consider subnet $T = S \circ M$.)

Base for a topology

Let $(X, \mathscr{T})$ be a topological space. A *base* for topology $\mathscr{T}$, $\mathscr{B}$, is a subset of $\mathscr{T}$ such that the set of arbitrary unions of elements of $\mathscr{B}$, $\{\bigcup \mathscr{C} | \mathscr{C} \subset \mathscr{B}\}$, equals $\mathscr{T}$. A *subbase* for topology $\mathscr{T}$, $\mathscr{S}$, is a subset of $\mathscr{T}$ such that the set of finite intersections of the members of $\mathscr{S}$, $\{\bigcap \mathscr{C} | \mathscr{C}$ is a finite subset of $\mathscr{S}\}$, is a base for topology $\mathscr{T}$. The concept of subbase (or base) for a topology is important because it characterizes such properties as minimal requirements in various topological arguments for a given topology. For example, we can see that net $S : D \to X$ in X converges to $x^* \in X$ if and only if for every neighborhood U of x^* belonging to a subbase for the topology, S is eventually in U.

Product topology

Consider family of sets $\{X_i | i \in I\}$. If each X_i is a topological space with topology $\mathscr{T}_i$, the *product topology* on $\prod_{i \in I} X_i$ is a topology whose subbase is the family that consists of set $\{f | f : I \to \bigcup_{i \in I} X_i, \forall i \in I \setminus \{j\}, f(i) \in X_i, f(j) \in U_j\}$ for some $j \in I$ and $U_j \in \mathscr{T}_j$. By considering the definition of subbase, product topology may be characterized as the weakest topology such that for every $j \in I$, the *projection* $\mathrm{pr}_j : \prod_{i \in I} X_i \ni (\cdots, x_j, \cdots) \mapsto x_j \in X_j$ is continuous. It can also be verified that net S in product space $\prod_{i \in I} X_i$ (the product set under the product topology) converges to x^* if and only if each net $\mathrm{pr}_j \circ S$ in j-th coordinate space, X_j, converges to the j-th coordinate $x_j^* = \mathrm{pr}_j(x^*)$ of x^*.

Quotient topology

Assume that $\mathscr{R}$ is an equivalence relation on topological space X. For each $x \in X$, denote by $[x]$ the equivalence class of x, i.e., $[x] = \{y \in X | y \mathscr{R} x\}$. The family of all such equivalence classes, $\{[x] | x \in X\}$, gives a *decomposition (partition)* of X, i.e., a disjoint family of subsets of X whose union is X. Decomposition $\{[x] | x \in X\}$, which is also denoted by $X/\mathscr{R}$, is called the *quotient set* of X with respect to $\mathscr{R}$. On X to $X/\mathscr{R}$, we may naturally define function $P : X \to X/\mathscr{R}$ to assign each $x \in X$ to its equivalence class $[x] \in X/\mathscr{R}$. P is called the *projection (quotient map)* of X onto quotient set $X/\mathscr{R}$. The *quotient topology* on quotient set $X/\mathscr{R}$ of topological space X is family $\{O | O \subset X/\mathscr{R}, P^{-1}(O) \text{is open}\}$, which is the finest topology such that quotient map $P : X \to X/\mathscr{R}$ is continuous.

Other concepts

For finite set A, we denote by $\sharp A$ the number of elements of A. The set of real numbers is denoted by R. We assume that readers have basic knowledge of the topological and algebraic features of R as a *conditionally complete ordered field*.[12] Denote by R_+ (resp., by R_{++}) the set of all non-negative reals (resp., strictly positive reals) and by R^n the n-th product of the set of real numbers. If there are no additional explanations, R^n is supposed to have the product of the usual (order) topology of R with vector-space, inner-product, and Euclidean-metric structures (n-dimensional Euclidean space). For easily understanding this book, the reader needs the most basic knowledge of Euclidean spaces.

1.3.2 *Compact sets, open coverings, and partition of unity*

Since the open *covering* of a space is an extremely important concept throughout this book, it is appropriate to use one subsection here to state several inherent concepts and properties that are repeatedly used in later chapters.

Let X be a topological space. A family of open subsets of X, $\{M_i | i \in I\}$, is said to be a *covering* of X if $\bigcup_{i \in I} M_i = X$. For two coverings, $\mathscr{M} = \{M_i | i \in I\}$ and $\mathscr{N} = \{N_j | j \in J\}$ of X, $\mathscr{N}$ is a *subcovering* (resp., *refinement*) of $\mathscr{M}$, if and only if for all $N_j \in \mathscr{N}$, there exists $M_i \in \mathscr{M}$ such that $N_j = M_i$ (resp., $N_j \subset M_i$). Covering $\mathscr{M}$ is said to be *finite* if $\mathscr{M}$ is a finite set.

[12] If unsure, see Debreu (1959).

Topological space X is *compact* if each covering of X has a finite subcovering. Equivalently, it can be said that space X is compact if and only if arbitrary non-empty family $\{F_i | i \in I\}$ of closed sets in X having the *finite intersection property* (every finite intersection among sets in $\{F_i | i \in I\}$ has a non-empty intersection) has a non-empty intersection. The compactness can also be characterized through the convergence of nets in the space.

THEOREM 1.3.1: (Net Characterization of Compactness) *Topological space X is compact iff every net in X has a converging subnet.*

PROOF: To see that every net $S : (D, \geq) \to X$ in compact set X has a converging subnet, use the finite intersection property of the family of closures of sets $A_m = \{S(n) | n \geq m\}$, $m \in D$. To see the sufficiency, suppose that every net in X has a converging subnet. Then for arbitrary family $\{F_i | i \in I\}$ of closed sets in X having the *finite intersection property*, if we consider a net on the set of finite subsets of I directed by inclusion as $S : \mathscr{F}(I) \ni A \mapsto S(A) \in \bigcap_{i \in A} F_i$, the limit point of a converging subnet of S is easily seen to belong to all F_i, $i \in I$. ■

In Euclidean n-space, a closed bounded set is compact. (The fact is known as the Heine–Borel covering theorem.)

In this book, we base many theorems on Brouwer's classical fixed-point theorem (Theorem 2.1.1) that may be applicable to all sets *homeomorphic* to a non-empty compact convex subset of Euclidean space R^n.[13] So it is useful to remember the next property on the homeomorphism between compact spaces. (Topological space X is said to be *Hausdorff* if for all $x, y \in X$, $x \neq y$, two open sets U_x and U_y exist such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.)

THEOREM 1.3.2: (Isomorphism Between Compact Sets) *A continuous bijection on compact space X to Hausdorff space Y is a homeomorphism.*

PROOF: Let $f : X \to Y$ be a continuous bijection. (Note that by the continuity of bijection f, Y is also compact and X is also a Hausdorff

[13] Topological spaces X and Y are said to be *homeomorphic* if continuous bijection $f : X \to Y$ exists such that f^{-1} is also continuous. (Function f is called a homeomorphism between X and Y.) One can prove that if X has the fixed-point property (i.e., every continuous mapping on the space to itself has a fixed point), space Y homeomorphic to X also has the fixed-point property.

space.) We have to show that f^{-1} is continuous. Consider net $\{y^\nu\}$ in Y that converges to $y^* \in Y$. Since X is compact, net $x^\nu = \{f^{-1}(y^\nu)\}$ in X has a subnet $\{f^{-1}(y^{\nu(\mu)})\}$ in X converging to point $x^* \in X$. Since f is continuous, $f(x^{\nu(\mu)})$ must converge to $f(x^*)$, so $f(x^*)$ is a cluster point of converging net $\{y^\nu\}$; i.e., $f(x^*)$ must equal y^* since Y is a Hausdorff space. It remains to be shown that net $\{f^{-1}(y^\nu)\}$ converges to x^*. The above argument ensures that every converging subnet of $\{f^{-1}(y^\nu)\}$ must converge to the same point, $x^* = f^{-1}(y^*)$. If $\{f^{-1}(x^\nu)\}$ does not converge to x^*, again by the compactness of X, net $\{f^{-1}(y^\nu)\}$ has a subnet that converges to a point different from x^*: a contradiction. ■

By definition, every closed subset of a compact space is obviously also compact under the relativized topology. One can also prove that compact subset X of topological space Y is closed if the topology of Y is Hausdorff.

Hausdorff space X is said to be *normal* if for any two closed subsets, A and B, such that $A \cap B = \emptyset$, there are two open sets, U_A and U_B, such that $U_A \supset A$, $U_B \supset B$ and $U_A \cap U_B = \emptyset$. From the definition, in normal space X, every open neighborhood U of $x \in X$ clearly includes closed neighborhood C of x. (Consider two closed sets, $X \setminus U$ and $\{x\}$.) It is also easy to prove that every compact Hausdorff space is normal.

THEOREM 1.3.3: (Partition of Unity) *Let X be a normal space, and let $\mathscr{U} = \{U_1, \ldots, U_n\}$ be a finite covering of X. It is known that a family of non-negative real valued continuous functions exists, $f_1 : X \to R_+, \ldots, f_n : X \to R_+$, such that $f_i(x) = 0$ for all $x \in X \setminus U_i$ for each i, and $\sum_{i=1}^n f_i(x) = 1$ for all $x \in X$.*

The family of functions stated in the above theorem is called *a partition of unity* on space X *subordinate to covering* $\mathscr{U}$. The theorem is an immediate consequence of the so-called Urysohn's Lemma on two closed subsets of a normal space.[14] A complete proof is given in Mathematical Appendix I.

1.3.3 *Vector space duality and hyperplane*

We denote by R^n_+ (resp., by R^n_{++}) the set $\{(x_1, \ldots, x_n) | x_1 \in R_+, \ldots, x_n \in R_+\}$ (resp., $\{(x_1, \ldots, x_n) | x_1 \in R_{++}, \ldots, x_n \in R_{++}\}$) in n-dimensional

[14] The proof of this theorem is easy when the topology of X is given through a metric as in the Euclidean spaces. Let $F_i(x)$ be the distance from x to $X \setminus U_i$ and define $f_i(x)$ as normalization $F_i(x)/\sum_{j=1}^n F_j(x)$ for each i and $x \in X$.

Euclidean space R^n. Readers are expected to have the most basic knowledge of vector-space structure in Euclidean spaces.

A *vector space* over real field R is a set L on which mapping $(x, y) \mapsto x + y$ on $L \times L$ to L, called *addition*, and mapping $(a, x) \mapsto ax$ on $R \times L$ to L, called *scalar multiplication*, are defined to satisfy the following axioms: (In the following, x, y, z and a, b are arbitrary elements of L and R, respectively.)

(1) $(x + y) + z = x + (y + z)$
(2) $x + y = y + x$
(3) $a(x + y) = ax + ay$
(4) $(a + b)x = ax + bx$
(5) $a(bx) = (ab)x$
(6) $\exists 0, 0 + x = x + 0 = x$
(7) $\forall x, \exists -x, x + (-x) = (-x) + x = 0$
(8) $1x = x$

Mapping f on vector space L over R to vector space M over R is *linear* if $f(ax + by) = af(x) + bf(y)$ for all $x \in L$, $y \in L$, and $a, b \in R$.

For m points $x^1, \ldots, x^m$ of vector space L over R and m scalars $a_1, \ldots, a_m$ in R, point $x = \sum_{i=1}^{m} a_i x^i$ in L is a *linear combination* (under coefficients $a_1, \ldots, a_m$) of points $x^1, \ldots, x^m$. Points $x^1, \ldots, x^m$ are *linearly independent* if $\sum_{i=1}^{m} a_i x^i = 0 \implies a_1 = 0, a_2 = 0, \ldots, a_m = 0$. In other words, points $x^1, \ldots, x^m$ are linearly independent if no x^i can be represented as a linear combination of other points. More generally, if subset A of L is such that no element x of A can be represented as a linear combination of other (finite) points in A, then set A of the points is *linearly independent.* If A is not linearly independent, it is *linearly dependent.*

Subset M of vector space L is a *linear subspace* of L if all additions between points in M and all scalar multiplications of points in M are also points in M. For subspace M of L, the subset of form $x + M = \{x + z \mid z \in M\}$ for some $x \in L$ is called an *affine subspace* of L. If A is a linearly independent subset of vector space L over R, the set of all linear combinations of points in A, $L(A)$, forms a subspace of L. Linearly independent subset A is called a *basis* (*Hamel basis*) of $L(A)$. Linear mapping on $L(A)$ is uniquely determined by the images of elements of the basis.

In vector space L over R, if m coefficients $a_1, \ldots, a_m$ for m points $x^1, \ldots, x^m$ belong to R_+ and satisfy $\sum_{i=1}^{m} a_i = 1$, *linear combination*

$\sum_{i=1}^{m} a_i x^i$ is called a *convex combination* (under coefficients $a_1, \ldots, a_m$) of points $x^1, \ldots, x^m$. Subset X of vector space L over R is *convex* if all convex combinations of two points in X are also elements of X. Given set A of vector space L, $\operatorname{co} A$ denotes the set of all convex combinations among points in A. One may prove that $\operatorname{co} A$ is the smallest convex set that includes A, which is also equal to the intersection of all convex sets that include A. (Use the fact that an arbitrary intersection of convex sets is also convex.[15])

A *topological vector space* over R is a vector space having a topology on which the addition and scalar multiplication are continuous. (Since $(a^\nu x^\mu + b^\eta y^\zeta) - (a^* x^* + b^* y^*) = (a^\nu x^\mu - a^* x^*) + (b^\eta y^\zeta - b^* y^*)$, one can verify that they are indeed jointly continuous.) Therefore, if $A = \{x_1, \ldots, x_\ell\}$ is a linearly independent subset of topological vector space over R, bijective linear mapping $f : R^\ell \ni (a_1, \ldots, a_\ell) \mapsto a_1 x_1 + \cdots + a_\ell x_\ell \in L(A)$ is continuous. A family of neighborhoods of $x \in X$, $\mathscr{U}$, such that for each neighborhood U_x of x, member U of family $\mathscr{U}$ included in U_x exists, is called a *neighborhood base* at x. Neighborhood base at $0 \in X$ is called a 0-neighborhood base. This concept is important since the topological features of a topological vector space are completely determined by a 0-neighborhood base. A *locally convex space* is a Hausdorff topological vector space with a 0-neighborhood base consisting of convex sets.

For vector space E, real valued linear function f is called a *(real) linear form* (or a linear functional) on E. The set of all real linear forms, E^*, may also be considered a vector space by defining $(f + g)(x)$ as $f(x) + g(x)$ and $(\alpha f)(x)$ as $\alpha f(x)$. E^* is called the *algebraic dual space* (or *algebraic dual*) of E. On topological vector space E, the set of all continuous real linear forms, E', is also recognized as a vector space and is called the *topological dual space* (or *topological dual*) of E.[16] The *weak topology* on E, $\sigma(E, E')$, is a topology whose subbase is constructed by sets of form $\{y \in E \mid f(y) < \alpha\}$ for some $f \in E'$ and $\alpha \in R$. It is the weakest locally convex topology under

[15] As stated in Section 1.2, although the "convexity" concept in this book is often used in the generalized sense, it may not be so harmful to give priority to the vector-space interpretation over the generalized one when a vector space structure is explicitly given.

[16] For example, let R^∞ be the set of the countably infinite product of R and let R_∞ be the subspace of R^∞ that consists of points whose coordinates are all 0 except for finite components. By considering the duality operation, $\langle (1, 1, \cdots), (x_1, \cdots, x_n, 0, \cdots, 0) \rangle = 1x_1 + \cdots + 1x_n$ for $(x_1, \cdots, x_n, 0, \cdots) \in R_\infty$, we can recognize $(1, 1, \cdots) \in R^\infty$ as an algebraic linear form on R_∞. The element $(1, 1, \cdots) \in R^\infty$ is not continuous, however, if we relativize the product topology on R^∞ to R_∞.

which every $f \in E'$ is continuous. On the other hand, the topology on E', whose subbase is constructed by sets of form $\{f \in E' | f(y) < \alpha\}$ for some $y \in E$ and $\alpha \in R$, is called the *weak star topology* on E', $\sigma(E', E)$.

If f is a real linear form on vector space E, set H of form $\{y \in E | f(y) = \alpha\}$ for some $\alpha \in R$ is called a *hyperplane* in E. In topological vector space E, hyperplane H is closed if and only if it is associated with continuous linear form f. We say that two sets, A and B, in vector space E are *separated* (resp., *strictly separated*) by a hyperplane if hyperplane $H = \{y | f(y) = \alpha\}$ exists such that $\forall a \in A, \forall b \in B$, $f(a) \leq \alpha \leq f(b)$, (resp., $f(a) < \alpha < f(b)$). The next theorem is especially critical for economic arguments. (For the proof, see Mathematical Appendix II. See also Schaefer (1971, p. 64, 9.1).)

THEOREM 1.3.4: (First Separation Theorem) *In topological vector space E, if A is a convex set whose interior* $\text{int}\, A$ *is non-empty and B is a non-empty convex set such that* $\text{int}\, A \cap B = \emptyset$, *then closed hyperplane H exists that separates A and B. If both A and B are open, we may choose H so that A and B are strictly separated.*

THEOREM 1.3.5: (Second Separation Theorem) *In locally convex space E, if A is a non-empty closed convex set and B is a non-empty compact convex set such that $A \cap B = \emptyset$, then a closed hyperplane exists that strictly separates A and B.*

PROOF: Under the basic property of vector space topology, set $-A = \{-a | a \in A\}$ is closed. Since B is compact, we can also verify that $B + (-A) = \{b + (-a) | b \in B, a \in A\}$ is closed. (Use a net and a converging subnet in compact set B.) Then, there is convex 0-neighborhood U that does not intersect with $B + (-A)$. (In the following, for subsets in a vector space, such notations as $A + B$, $-A$ and $B + (-A) = B - A$ will be used without any explanations. If one such set is a singleton, we often write $x + A$ instead of $\{x\} + A$.) Without loss of generality, we may assume U to be open. (Note that the interior of a convex set is always open under vector space topology.) Let $W = U \cap -U$ and define V as $V = (1/3)W = \{(1/3)w | w \in W\}$. Then, $A + V$ and $B + V$ are two disjoint convex open sets satisfying all conditions in Theorem 1.3.4, and thus the result is an immediate consequence of the First Separation Theorem. ■

Notes on References

Since this is a research monograph, many theorems and arguments must be supplemented with sources to establish priority or confidence. At the same time, I want this book to be readable as a text for graduate students in economics who are concerned with rigorous mathematical arguments. Therefore, in the main sections of this book, references to the literature for every important (especially mathematical) theorem and concept have been minimized as suggestions for further reading from an educational viewpoint. References necessary for research-level arguments are given in the last section of each chapter as Bibliographic Notes.